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# Homomorphism and quotient of fuzzy $k$-hyperideals

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**Abstract**

In [15], we introduced the notion of weak (resp. strong) fuzzy $k$-hyperideal. In this note we investigate the behavior of them under homomorphisms of semihyperrings. Also we define the quotient of fuzzy weak (resp. strong) $k$-hyperideals by a regular relation of semihyperring and obtain some results.

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# 1 Introduction

Following the introduction of fuzzy set by L. A. Zadeh in 1965 ([26]), the fuzzy set theory developed by Zadeh himself and can be found in mathematics and many applied areas. The concept of a fuzzy group was introduced by A. Rosenfeld in [24]. The notion of fuzzy ideals in a ring was introduced and studied by W. J. Liu [20]. T.K. Dutta and B. K. Biswas studied fuzzy ideals, fuzzy prime ideals of semirings in [14, 16] and they defined fuzzy ideals of semirings and fuzzy prime ideals of semirings and characterized fuzzy prime ideals of non-negative prime integers and determined all it's prime ideals. Recently, Y. B. Jun, J. Neggeres and H. S. Kim ([16]) extended the concept of a L-fuzzy (characteristic) ideal left(resp. right) ideal of a ring to a semiring. S. I. Baik and H. S. Kim introduced the notion of fuzzy $k$-ideals in semirings [6].

Also a hypergroup was introduced by F. Marty ([23]), today the literature on hypergroups and related structures counts 400 odd items [8, 9, 25]. Among the several contexts which they aries is hyperrings. First M. Krasner studied hyperrings, which is a triple $(R, +, .)$, where $(R, +)$ is a canonical hypergroup and $(R, .)$ is a semigroup, such that for all $a, b, c \in R$, $a(b + c) = ab + ac, (b + c)a = ba + ca$ ([18]). Zahedi and others introduced and studied the notion of fuzzy hyperalgebraic structures [3, 4, 5, 11, 12, 19, 27]. In [15] we introduced the notion of fuzzy weak (strong) $k$-hyperideal and then we obtained some related basic results. In this note we investigate the behavior of them under homomorphisms of semihyperrings. Also we define the quotient of fuzzy weak (strong) $k$-hyperideals by a regular relation of semihyperring and obtain some results.

# 2 Preliminaries

In this section we gather all definitions and simple properties we require of semihyperrings and fuzzy subsets and set the notions.

A map $\circ : H \times H \longrightarrow P_*(H)$ is called *hyperoperation* or *join operation.*

A *hypergroupoid* is a set $H$ with together a (binary) hyperoperation $\circ$. A hypergroupoid $(H, \circ)$, which is associative, that is $x \circ (y \circ z) = (x \circ y) \circ z, \forall x, y, z \in H$ is called a *semi-hypergroup* . A *hypergroup* is a semihypergroup such that $\forall x \in H$ we have $x \circ H = H = H \circ x$, which is called *reproduction* axiom.

Let $H$ be a hypergroup and $K$ a nonempty subset of $H$. Then $K$ is a *subhypergroup* of $H$ if itself is a hypergroup under hyperoperation restricted to $K$. Hence it is clear that a subset $K$ of $H$ is a subhypergroup if and only if $aK = Ka = K$, under the hyperoperation on $H$.

A set $H$ together a hyperoperation $\circ$ is called a *polygroup* if the following conditions are satisfied:

(1) $(x \circ y) \circ z = x \circ (y \circ z) \ \ \forall x, y, z \in H$;

(2) $\exists e \in H$ as unique element such that $e \circ x = x = x \circ e \ \ \forall x \in H$;

(3) $\forall x \in H$ there exists an unique element, say $x' \in H$ such that $e \in x \circ x' \cap x' \circ x$ ( we denote $x'$ by $x^{-1}$).

(4) $\forall x, y, z \in H, \ z \in x \circ y \Longrightarrow x \in z \circ y^{-1} \Longrightarrow y \in x^{-1} \circ z.$

A non-empty subset $K$ of a polygroup $(H, \circ)$ is called a *subpolygroup* if $(K, \circ)$ is itself a polygroup. In this case we write $K <_P H$.

A commutative polygroup is called *canonical hypergroup.*

**Definition 2.1**. A hyperalgebra $(R, +, .)$ is called a semihyperring if and only if

$(i)$ $(R, +)$ is a semihypergroup and $(R, .)$ is a semigroup;

$(ii)$ $a.(a + b) = a.b + a.c$ and $(b + c).a = b.c + c.a$ $\forall a, b, c \in R$.

A semihyperring is called with zero, if there exists an element, say $0 \in R$ such that $0.x = 0 = x.0$ and $0 + x = x = x + 0$ $\forall x \in R$.

Also a semihyperring $(R, +, .)$ is called a *hyperring* provided $(R, +)$ is a canonical hypergroup.

A hyperring $(R, +, .)$ is called

$(i)$ *commutative* if and only if $a.b = b.a$ $\forall a, b \in R$;

$(ii)$*with identity*, if there exists an element, say $1 \in R$ such that $1.x = x.1 = x$ $\forall x \in R$.

Let $(R, +, .)$ be a hyperring, a nonempty subset $S$ of $R$ is called a subhyperring of $R$ if $(S, +, .)$ is itself a hyperring.

**Definition 2.2**. A subhyperring $I$ of a hyperring $R$ is a *(resp. left) right hyperideal* of $R$ provided that ( resp. $x.r \in I$) $r.x \in I$ $\forall r \in R$, $\forall x \in I$. $I$ is called a *hyperideal* if $I$ is both left and right hyperideal.

We use $I = [0, 1]$, the real unit interval as a chain with the usual ordering, in which $\bigwedge$ stands for infimum (inf ) (or intersection ) and $\bigvee$ stands for supremum ( sup ) (or union), for the degree of membership.

A *fuzzy subset* of a given set $X$ is a mapping $\mu : X \longrightarrow I$. We denote the set of all fuzzy subsets of $X$ by $FS(X)$. For $\mu \in FS(X)$, the level subset of $\mu$ is defined by

$$\mu_t = \{x \in X |\ \mu(x) \geq t\} \quad \forall t \in I.$$

For a fuzzy subset $\mu$ of $X$ we denote by $Im(\mu)$ the image of $\mu$. Let $\{\mu_i\ |$

$i \in I\}$ be a family of fuzzy subsets, *intersection* of $\mu_i$'s is defined by

$$(\bigcap_{i\in I} \mu_i)(x) = \bigwedge_{i\in I} \mu_i(x).$$

**Definition 2.3**. Let $(G, .)$ be a group and $\mu \in FS(G)$. Then $\mu$ is said to be a *fuzzy subgroup* of $G$ if $\forall x, y \in G$ we have :

(*i*) $\mu(xy) \geq \mu(x) \wedge \mu(y)$;

(*ii*) $\mu(x^{-1}) \geq \mu(x)$.

**Definition 2.4**. If $f : X \longrightarrow Y$ be a function and $\mu \in FS(X)$ , then we say $\mu$ is $f-$invariant if and only if

$$f(a) = f(b) \Longrightarrow \mu(a) = \mu(b).$$

In the sequel by $R$ we mean a semihyperring.

**Definition 2.5**.[1] A nonempty subset $I$ of $R$ is called

(*i*) a *left (resp. right) hyperideal* of $R$ if and only if

(1) $(I, +)$ is a semihypergroup of $(R, +)$;

(2) $rx \in I$ (resp. $xr \in I$), for all $r \in R$ and for all $x \in I$.

(*ii*) a *hyperideal* of $R$ if it is both a left and a right hyperideal of $R$. By $I <_h R$, we mean hyperideal of $R$.

(*iii*) a left hyperideal $I$ of $R$ is called *weak left k-hyperideal* of $R$ if for $a \in I$ and $x \in R$ we have

$$a + x \subseteq I \ \text{ or } \ x + a \subseteq I \quad \Longrightarrow \quad x \in I.$$

A left hyperideal $I$ of $R$ is called *strong left k-hyperideal* of $R$ if for $a \in I$ and $x \in R$ we have

$$a + x \approx I \ \text{ or } \ x + a \approx I \quad \Longrightarrow \quad x \in I,$$

where by $A \approx B$, we mean $A \cap B \neq \emptyset$, for all nonempty subsets $A$ and $B$ of $R$.

A right(resp. strong) weak $k$-hyperideal is defined dually. A two sided (resp. strong) weak $k$-hyperileal or simply a (resp. strong) weak $k$-hyperideal is both left and right (resp. strong) weak $k$-hyperideal. We denote $I <_{w.k.h} R$ (resp. $I <_{s.k.h} R$) for weak (resp. strong) $k$-hyperideal of $R$.

Clearly, every (strong) weak $k-$ hyperideal is a hyperideal, but the converse is not true.

**Example**. Consider $\mathbb{Z}$, the set of integer numbers. Define new hyperoperations $\oplus$ and $\circ$ on $\mathbb{Z}$ as follow

$$m \oplus n = \{m, n\} \qquad \text{and} \qquad m \circ n = mn \quad \forall m, n \in \mathbb{Z}.$$

Clearly $(\mathbb{Z}, \oplus, \circ)$ is a semihyperring. Now it is easy to verify that $I =< 2 >= \{2k \mid k \in \mathbb{Z}\}$, is a hyperideal of $\mathbb{Z}$, but it isn't strong $k-$hyperideal, since $3 \oplus 2 = \{3, 2\} \approx I$ and $2 \in I$ but $3 \notin I$.

**Definition 2.6** .[7] Let $R$ and $S$ be semihyperrings. A mapping $f : R \longrightarrow S$ is said to be

$(i)$ *homomorphism* if and only if

$f(x + y) \subseteq f(x) + f(y)$ and

$f(x.y) = f(x).f(y) \ \ \forall x, y \in R.$

$(ii)$ *good homomorphism* if and only if

$f(x + y) = f(x) + f(y)$ and

$f(x.y) = f(x).f(y) \ \ \forall x, y \in R.$

**Definition 2.7** .[15] A fuzzy subset $\mu$ of a semihyperring $R$ is called a *fuzzy* *left hyperideal* of $R$ if and only if

$(i)$ $\bigwedge_{z\in x+y} \mu(z) \geq \mu(x) \bigwedge \mu(y) \quad \forall x, y \in R;$

$(ii)$ $\mu(xy) \geq \mu(y) \quad \forall x, y \in R.$

A fuzzy right hyperideal is defined dually. A fuzzy left and right hyperideal is called a fuzzy hyperideal. We denote $\mu <_{f.h} R$ for fuzzy hyperideal of $R$.

**Definition 2.8**.[15] A fuzzy hyperideal $\mu$ of $R$ is called

$(i)$ a *weak fuzzy $k$-hyperideal* of $R$ if and only if

$$\mu(x) \geq [(\bigwedge_{u\in x+y} \mu(u)) \bigvee (\bigwedge_{v\in y+x} \mu(v))] \bigwedge \mu(y) \qquad \forall x, y \in R.$$

$(ii)$ a *strong fuzzy $k$-hyperideal* of $R$ if and only if

$$\mu(x) \geq (\mu(z) \vee \mu(z^{'})) \wedge \mu(y) \qquad \forall z \in x + y, \forall z^{'} \in y + x.$$

Note that if $(R, +)$ is a commutative semihyperring, then the above conditions reduce to the following conditions:

$$\mu(x) \geq (\bigwedge_{u\in x+y} \mu(u)) \bigwedge \mu(y) \qquad \forall x, y \in R.$$

and

$$\mu(x) \geq \mu(z) \wedge \mu(y) \qquad \forall z \in x + y.$$

We denote by $\mu <_{w.f.k.h} R$ (resp. $\mu <_{s.f.k.h} R$), for a weak fuzzy $k-$hyperideal (resp. strong fuzzy $k-$hyperideal) of $R$.

**Proposition 2.9**.[15] Let $R$ be a semihyperring and $\mu \in FS(R)$. Then

$(i)$ $\mu$ is a fuzzy hyperideal of $R$ if and only if every nonempty level subset, $\mu_t$ is a hyperideal of $R$.

$(ii)$ $\mu$ is a weak fuzzy $k$-hyperideal of $R$ if and only if every nonempty level subset, $\mu_t$ is a weak $k$-hyperideal of $R$.

$(iii)$ $\mu$ is a strong fuzzy $k$-hyperideal of $R$ if and only if every nonempty level subset, $\mu_t$ is a strong $k$-hyperideal of $R$.

**Lemma 2.10**. Let $R$ be a semihyperring with zero and $\mu$ be a fuzzy hyperideal of $R$. Then $\mu(x) \leq \mu(0) \quad$ for all $x \in R$.

# 3 Homomorphisms of Fuzzy $k$-Hyperideals

In this section we investigate the behavior of fuzzy weak (strong) $k$-hyperideals under homomorphisms of semihyperrings.

**proposition 3.1**. Let $f : R \longrightarrow R^{'}$ be a homomorphism of semihyperrings. If $\nu <_{s.f.k.h} R'$, then $f^{-1}(\nu) <_{s.f.k.h} R$.

**proof**. We know that $f^{-1}(\nu)(x) = \nu(f(x))$. Let $x,\ y \in R$ and $z \in x + y$, then we have $f(z) \in f(x + y) \subseteq f(x) + f(y)$, and since $\nu <_{f.h} R'$, it concluded that $\nu(f(z)) \geq \nu(f(x)) \wedge \nu(f(y))$.

Also

$$\nu(f(xy)) = \nu(f(x)f(y)) \geq \nu(f(x)) \vee \nu(f(y)).$$

Therefore $f^{-1}(\nu) <_{f.h} R$.

Now let $z \in x + y$ and $z' \in y + x$, thus $f(z) \in f(x) + f(y)$ and $f(z') \in f(y) + f(x)$, then $\nu <_{s.f.k.h} R'$ implies that

$$\nu(f(x)) \geq [\nu(f(z)) \vee \nu(f(z'))] \wedge \nu(f(y))$$

as required.

**proposition 3.2**. Let $f : R \longrightarrow R^{'}$ be a good homomorphism of semihyperrings. If $\nu <_{w.f.k.h} R^{'}$, then $f^{-1}(\nu) <_{w.f.k.h} R$.

**proof**. We know that $f^{-1}(\nu)(x) = \nu(f(x))$. First we prove that $f^{-1}(\nu)$ is a fuzzy hyperideal of $R$. Let $x,\ y \in R$ and $z \in x + y$, we should prove that

$$\nu(f(z)) \geq \nu(f(x)) \wedge \nu(f(y)) \qquad (1)$$

(1) is valid because $\nu$ is a fuzzy hyperideal and $f$ is a good homomorphism, then for $z \in x + y$ we have $f(z) \in f(x + y) = f(x) + f(y)$.

Also similar previous proposition

$$\nu(f(xy)) \geq \nu(f(x)) \vee \nu(f(y)).$$

Therefore $f^{-1}(\nu) <_{f.h} R$.

Now we prove that $f^{-1}(\nu) <_{w.f.k.h} R$, that is

$$f^{-1}(\nu)(x) \geq \{(\bigwedge_{t \in x+y} f^{-1}(\nu)(t)) \bigvee (\bigwedge_{t^{'} \in y+x} f^{-1}(\nu)(t^{'}))\} \bigwedge f^{-1}(\nu)(y) \qquad (2).$$

Note that since $f$ is a good homomorphism, then $t \in x + y$ if and only if $f(t) \in f(x) + f(y)$, and also $\nu <_{w.f.k.h} R^{'}$, we have

$$\nu(f(x)) \geq \{(\bigwedge_{f(t) \in f(x)+f(y)} \nu(f(t))) \bigvee (\bigwedge_{f(t^{'}) \in f(y)+f(x)} \nu(f(t^{'})))\} \bigwedge \nu(f(y)).$$

The last relation implies (2), and this complete the proof.

**proposition 3.3**. Let $f : R \longrightarrow R^{'}$ be a good epimorphism of semihyperrings. If $\mu <_{w.f.k.h} R$ (resp. $\mu <_{s.f.k.h} R$) and $\mu$ be $f-$invariant, then $f(\mu) <_{w.f.k.h} R^{'}$ (resp. $f(\mu) <_{s.f.k.h} R$).

**proof**. First we show that $f(\mu) <_{f.h} R^{'}$.

Let $a,\ b \in R^{'}$ and $c \in a+b$, we should prove that

$$f(\mu(c)) \geq f(\mu(a)) \wedge f(\mu(b)).$$

We have

$$\begin{aligned} f(\mu(c)) &= \bigvee_{z \in f^{-1}(c)} \mu(z), \\ f(\mu(a)) &= \bigvee_{x \in f^{-1}(a)} \mu(x), \\ f(\mu(b)) &= \bigvee_{y \in f^{-1}(b)} \mu(y). \end{aligned}$$

Since $\mu$ is $f-$invariant, then

$$\exists z_0 \in f^{-1}(c),\ f(\mu(c)) = \mu(z_0),$$

$$\exists x_0 \in f^{-1}(a),\ f(\mu(a)) = \mu(x_0),$$

$$\exists y_0 \in f^{-1}(b),\ f(\mu(b)) = \mu(y_0),$$

therefore

$$\begin{aligned} f(z_0) = c,\ f(x_0) = a,\ f(y_0) = b &\Longrightarrow f(z_0) \in f(x_0) + f(y_0) \\ &\Longrightarrow z_0 \in x_0 + y_0 \qquad \text{( f is a good homomorphism )} \\ &\Longrightarrow \mu(z_0) \geq \mu(x_0) \wedge \mu(y_0) \qquad (\mu <_{f.h} R) \\ &\Longrightarrow f(\mu(c)) \geq f(\mu(a)) \wedge f(\mu(b)). \end{aligned}$$

For proving the second condition of a fuzzy hyperideal, we should prove that

$$f(\mu)(r^{'}x^{'}) \geq f(\mu)(x^{'}) \vee f(\mu)(r^{'}) \quad \forall\ \ r^{'}, x^{'} \in R^{'}$$

Since $f$ is onto, then $r' = f(r)$ and $x' = f(x)$ for some $r$ and $x$ in $R$, Thus

$$\begin{aligned} f(\mu)(r^{'}x^{'}) &= \bigvee_{rx \in f^{-1}(r^{'}x^{'})} \mu(rx) \\ &= \mu(r_0x_0) \quad \exists r_0 \in f^{-1}(r'), x_0 \in f^{-1}(x') \quad (\mu \text{ is } f\text{-invariant}) \\ &\geq \mu(x_0) \vee \mu(r_0) \qquad\qquad (\ \mu <_{f.h} R) \\ &= f(\mu)(x^{'}) \vee f(\mu)(r^{'}) \quad (\mu \text{ is } f\text{-invariant}). \end{aligned}$$

Therefore

$$f(\mu)(r'x') \geq f(\mu)(r') \vee f(\mu)(x').$$

Now we prove that $f(\mu) <_{w.f.k.h} R^{'}$. Let $a,\ b \in R$, we show that

$$f(\mu)(a) \geq [(\bigwedge_{t \in a+b} f(\mu)(t)) \bigvee (\bigwedge_{t^{'} \in b+a} f(\mu)(t^{'}))] \bigwedge f(\mu)(b) \qquad (1)$$

Since $f$ is onto and $\mu$ is $f-$invariant, then

$$f(\mu)(a) = \mu(x_0),\ f(\mu)(t) = \mu(z_0),\ f(\mu)(t^{'}) = \mu(z_0^{'}),\ f(\mu)(b) = \mu(y_0),$$

where

$$x_0 \in f^{-1}(a),\ y_0 \in f^{-1}(b),\ z_0 \in f^{-1}(t),\ z^{'} \in f^{-1}(t^{'}).$$

Hence (1) reduced to the form

$$\mu(x_0) \geq [(\bigwedge_{t \in a+b} \mu(z_0)) \bigvee (\bigwedge_{t^{'} \in b+a} \mu(z_0^{'}))] \bigwedge \mu(y_0) \qquad (2)$$

On the other hand from above discussion and since $f$ is a good homomorphism $t \in a+b$ if and only if $f(z_0) \in f(x_0) + f(y_0)$ if and only if $z_0 \in x_0 + y_0$. Similarly, $t' \in b+a$ if and only if $z_0' \in y_0 + x_0$.

Therefore by (2), it is enough that we prove that

$$\mu(x_0) \geq [(\bigwedge_{z_0 \in x_0+y_0} \mu(z_0)) \bigvee (\bigwedge_{z_0^{'} \in y_0+x_0} \mu(z_0^{'}))] \bigwedge \mu(y_0),$$

but clearly the last statement is true, since $\mu <_{w.f.k.h} R$. This complete the proof.

In this part we define the quotient of fuzzy weak (strong) $k$-hyperideals by a regular relation of semihyperring

Let $R$ be a semihyperring and $\theta$ be an equivalence relation on $R$. Naturally we can extend $\theta$ to $\overline{\theta}$ to the subsets of $R$ as follow:

Let $A, B$ be nonempty subsets of $R$. Define

$$A\overline{\theta}B \Longleftrightarrow \quad \forall a \in A \;\; \exists b \in B : \quad a\theta b, \;\; \forall b \in B \;\; \exists \;\; a \in A : \quad b\theta a.$$

An equivalence relation $\theta$ on $R$ is said to be *regular* if for all $a,\ b,\ x \in R$ we have

$(i)$ $a\theta b \Longrightarrow (a+x)\overline{\theta}(b+x)$ and $(x+a)\overline{\theta}(x+b)$,

$(ii)$ $a\theta b \Longrightarrow (ax)\theta(bx)$ and $(xa)\theta(bx)$.

By $R:\theta$ we mean the set of all equivalence classes with respect to $\theta$, that is

$$R:\theta = \{r_\theta | r \in R\}.$$

**Remark 3.4**. We know that if $R$ is a semihyperring and $\theta$ is a regular equivalence relation on $R$ , then $R:\theta$ by hyperoperations $\oplus$ and $\odot$ is defined as follow

$$x_\theta \oplus y_\theta = \{x_\theta | z \in x+y\},$$

$$x_\theta \odot y_\theta = (xy)_\theta.$$

is a semihyperring. For $\mu \in FS(R)$, define $(\mu : \theta)(x_\theta) = \bigvee_{y \in x_\theta} \mu(y)$. Also we know that the mapping $\varphi : R \longrightarrow R : \theta$ defined by $\varphi(a) = a_\theta$ is a good epimorphism. Now if $\mu <_{w.f.k.h} R$ and $\mu$ be $\varphi-$invariant then by proposition 3.3 it concludes that $\varphi(\mu) = \mu : \theta <_{w.f.k.h} R : \theta$ .

**Proposition 3.5**. If $\mu <_{w.f.k.h} R$ and $R$ has zero, then $\mu_* = \{x \in R \mid \mu(x) = \mu(0)\}$ is a weak $k-$hyperideal of R.

**Proof**. First we prove that $\mu_* <_h R$. For $x,\ y \in \mu_*$ and $z \in x + y$, then $\mu(z) \geq \mu(x) \wedge \mu(y) = \mu(0)$, hence by Lemma 2.10 $\mu(z) = \mu(0)$, therefore $z \in \mu_*$.

Let $r \in R$ and $x \in \mu_*$, then we have

$$\begin{aligned}
\mu(rx) &\geq \mu(r) \vee \mu(x) \\
&= \mu(r) \vee \mu(0) \qquad (\ x \in \mu_*\ ) \\
&= \mu(0) \qquad (\text{ by Lemma 2.10 }) \\
&\Longrightarrow \mu(rx) = \mu(0) \qquad (\text{ by Lemma 2.10 }) \\
&\Longrightarrow rx \in \mu_*.
\end{aligned}$$

Now suppose $r + x \subseteq \mu_*$ or $x + r \subseteq \mu_*$ and $x \in \mu_*$, we show that $r \in \mu_*$.

From $\mu <_{w.f.k.h} R$ then we have :

$$\mu(r) \geq [(\bigwedge_{z \in r+x} \mu(z)) \bigvee (\bigwedge_{z' \in x+r} \mu(z'))] \bigwedge \mu(x).$$

Since $\mu(x) = \mu(0)$ and $\bigwedge_{z \in r+x} \mu(z) = \mu(0)$ and $\bigwedge_{z' \in x+r} \mu(z') = \mu(0)$, then $\mu(r) \geq \mu(0)$, and then by Lemma 2.10, $\mu(r) = \mu(0)$. Therefore $\mu_* <_{w.k.h} R$.

**Proposition 3.6**. If $\mu <_{s.f.k.h} R$, then $\mu^* = \{x \in R \mid \mu(x) > 0\}$ is a strong $k$-hyperideal of $R$.

**Proof**. Let $x,\ y \in \mu^*$ and $z \in x+y$, then by hypothesis yields

$$\mu(z) \geq \mu(x) \wedge \mu(y) > 0,$$

thus $z \in \mu^*$.

If $r \in R$ and $x \in \mu^*$, then we have

$$\mu(rx) \geq \mu(r) \vee \mu(x) \geq \mu(x) > 0,$$

therefore $rx \in \mu^*$. Similarly $xr \in \mu^*$. Thus $\mu^* <_h R$.

Now if $r + x \approx \mu^*$ or $x + r \approx \mu^*$ and $x \in \mu^*$.

By hypothesis we have

$$\mu(r) \geq (\mu(z) \vee \mu(z')) \wedge \mu(x) > 0 \qquad \forall z \in r+x \approx \mu^*,\ \forall z' \in x+r \approx \mu^*,$$

that is $r \in \mu^*$, and hence $\mu^* <_{s.k.h} R$.

**Proposition 3.7**. Let $R$ be a semihyperring with zero and $x, y \in R$:

$(i)$ If $\mu <_{w.f.k.h} R$ and $\mu(t) = \mu(0) = \mu(t')$ for all $t \in x+y$ and $t' \in y+x$, then $\mu(x) = \mu(y)$.

$(ii)$ If $\mu <_{s.f.k.h} R$ and $\mu(u) = \mu(0) = \mu(v)$ for some $u \in x+y$ and $v \in y+x$, then $\mu(x) = \mu(y)$.

**Proof**. $(i)$ Since $\mu <_{w.f.k.h} R$ and $\mu(t) = \mu(0) = \mu(t')$ for all $t \in x+y$ and $t' \in y+x$, then $\bigwedge\limits_{t \in x+y} \mu(t) = \mu(0) = \bigwedge\limits_{t' \in y+x} \mu(t')$, thus

$$\begin{aligned}
\mu(x) &\geq [(\bigwedge_{t \in x+y} \mu(t)) \bigvee (\bigwedge_{t' \in y+x} \mu(t'))] \bigwedge \mu(y) \\
&= \mu(0) \wedge \mu(y) \\
&= \mu(y) \qquad\qquad \text{(by Lemma 2.10)} \\
&\Longrightarrow \mu(x) \geq \mu(y).
\end{aligned}$$

Similarly we conclude that $\mu(y) \geq \mu(x)$. Therefore $\mu(x) = \mu(y)$.

$(ii)$ Suppose $u \in x + y$ and $v \in y + x$ such that $\mu(u) = \mu(0) = \mu(v)$, since $\mu <_{s.f.k.h} R$, then

$$\begin{aligned} \mu(y) \geq (\mu(u) \vee \mu(v)) \wedge \mu(x) &= \mu(0) \wedge \mu(x) \quad (\text{ by hypothesis }) \\ &= \mu(x) \qquad\qquad (\text{by Lemma 2.10}) \\ &\Longrightarrow \mu(y) \geq \mu(x). \end{aligned}$$

Similarly we obtain $\mu(x) \geq \mu(y)$. Therefore $\mu(x) = \mu(y)$.

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